

Outer space and Automorphisms of free groups

LECTURE 5

Last time we defined the spine of Outer space and used it to get algebraic info about $\text{Out} F_n$ (cohomology)

Today - take a closer look at the spine

$K_n =$ spine of outer space

= subcomplex of barycentric subdivision $S'(M_n)$

of the sphere complex spanned by simple systems.

- contractible

- $\dim = 2n - 3$

- $\text{Out}(F_n)$ acts properly, simplicially

- Quotient is compact

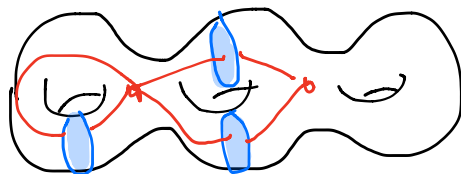
In particular, $\text{Out}(F_n)$ is quasi-isometric to K_n

So q.i. invariants of K_n are

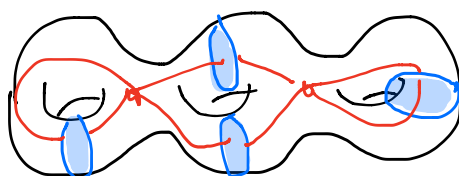
q.i. invariants of $\text{Out}(F_n)$

Description of K_n in terms of marked graphs

Sphere system $\mathcal{A} = \{\Delta_0, \dots, \Delta_k\} \leftrightarrow$ dual graph G



\mathcal{A} is simple $\iff G \hookrightarrow M$ induces
 $\pi_1 G \xrightarrow{\cong} \pi_1 M$



So if we identify $\pi_1 M \cong F_n$ the embedding provides a

marking, ie $\mathcal{A} \leftrightarrow (g, G) = \text{marked graph}$

(can also make the marking go the other direction)

Edges in K_n are inclusions $\delta_0 \subseteq \delta_1$

Digression: poset terminology

X a simplicial complex X' its barycentric subdivision is an example of a geometric realization of a partially ordered set (\equiv poset)

P a poset $\rightsquigarrow |P|$ is a simplicial complex

with vertices = elements x of P

k -simplices = chains $x_0 \leq x_1 \leq \dots \leq x_k$

X' = realization of poset of simplices ordered by inclusion.

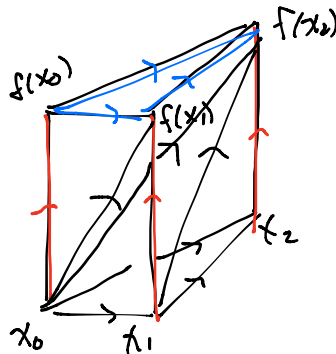
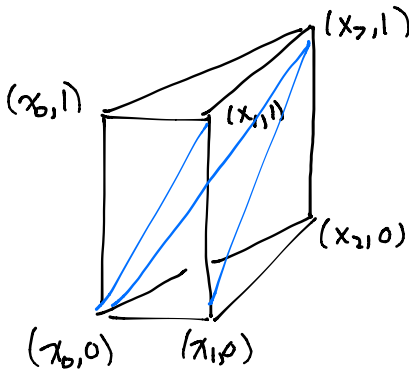
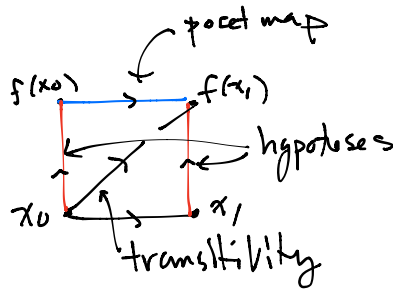
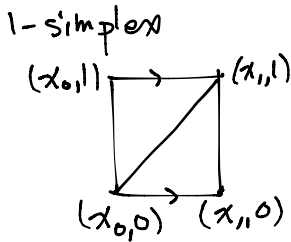
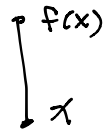
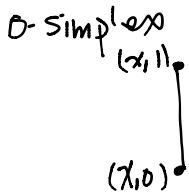
A poset map $f: X \rightarrow Y$ is a map respecting the partial ordering, i.e. $x \leq x' \Rightarrow f(x) \leq f(x')$

Useful Lemma

$f: X \rightarrow X$ a poset map
with $f(x) \geq x$ for all x . Then $|X| \simeq |f(X)|$.

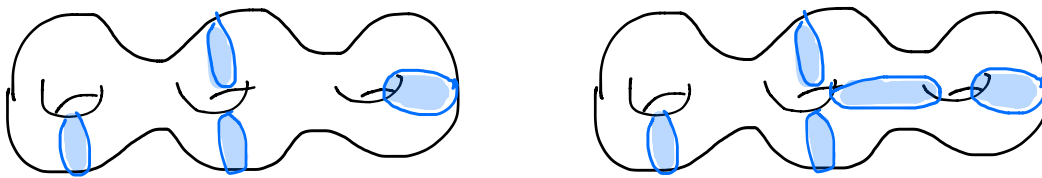
Proof Want $H: |X| \times I \rightarrow |X|$ with
 $H(x, 0) = x, H(x, 1) = f(x)$.

Use "prism operator" to triangulate
 $\sigma \times I, \sigma$ a simplex

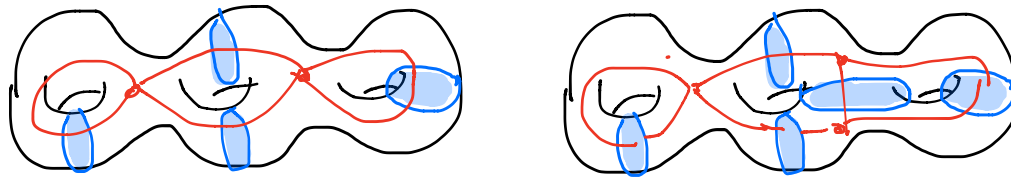


Note you can replace $f(x) \geq x$ with $f(x) \leq x$
 (but the picture is upside down...)

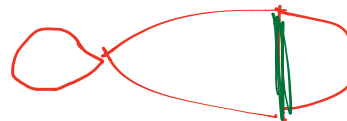
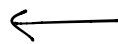
Back to spheres and graphs
 edge in K_n is inclusion $\Delta_0 \subset \Delta_1$



effect on dual graphs:



collapse an edge



expand a vertex into 2 vertices + edge

you can collapse many edges as long as they don't contain a cycle

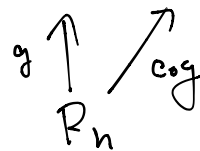
Def a forest is a subgraph with no cycles

A Forest collapse collapses each edge in the forest to a point (=vertex)

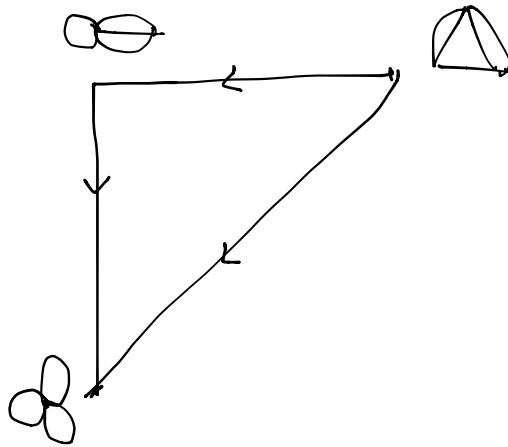
Graphical def'n of K_n : Geometric realization of poset of marked graphs (G, g) , where

$(G, g) \geq (G', g')$ if \exists forest collapse $G \rightarrow G'$

making diagram commute :



Example of a 2-simplex



Local structure of K_n :

Link of a vertex Δ :

Two types of edges connected to Δ

$$\Delta' < \Delta \qquad \Delta'' > \Delta$$

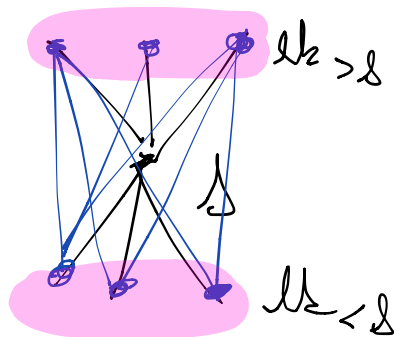
"lower link" "upper link"

Everything in lower link is connected to

everything in upper link, so

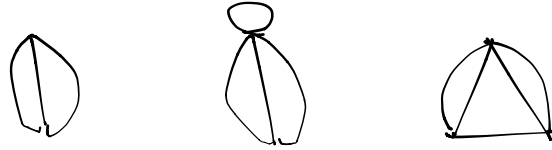
The link is the simplicial join of lower and upper links

$$lk(\Delta) = lk_{<\Delta} * lk_{>\Delta}$$



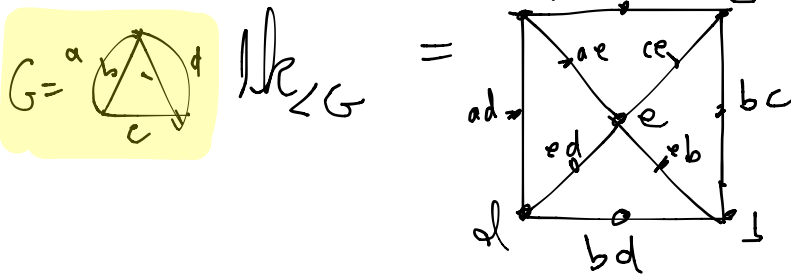
$lk_{\leq \lambda}$. Use graph description of $lk_{\leq \lambda}$

Examples



$G = \begin{matrix} a \\ \circ \\ b \\ \circ \\ c \end{matrix}$ $lk_{\leq G} = \begin{matrix} \circ & \circ & \circ \\ G_a & G_b & G_c \end{matrix} \equiv \begin{matrix} a & b & c \end{matrix}$

$G = \begin{matrix} a \\ \circ \\ b \\ \circ \\ c \end{matrix}$ $lk_{\leq G} = \begin{matrix} a & b & c \\ \circ & \circ & \circ \end{matrix}$



In general, $lk_{\leq \lambda} = \begin{cases} \text{geom. realization of poset} \\ F(G) \text{ of forests in} \\ \text{the dual graph } G. \end{cases}$

Thm If G is connected then $|F(G)|$ is either contractible or homotopy equivalent to a wedge of spheres $\bigvee S^{V-2}$, where $V = \# \text{ vertices of } G$.
 $|F(G)|$ is contractible iff G has a separating edge.

Proof Induct on $V+E$. ($E = \# \text{edges of } G$)

Let e be an edge of G

If e is a loop, then $F(G) = F(G-e)$, so done
by induction

If e is a bridge (= separating edge) then
you can add e to any forest Φ , so

$\Phi \mapsto \Phi \cup e \rightarrow e$ are poset maps

contracting $|F(G)|$ to $|e|$

(So we may assume G has no bridges)

Then $G-e$ is connected so Thm is true for
 $G-e$.

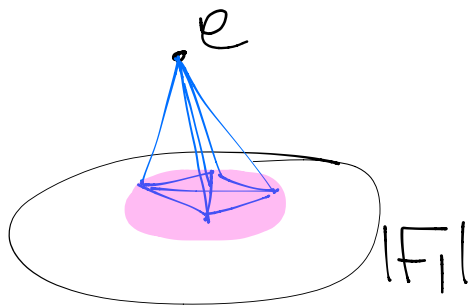
Let $F_i \subset F(G)$ be all forests except $\{e\}$

$F_i \rightarrow F(G-e)$

is a poset map giving

$\phi \mapsto \begin{cases} \phi & e \notin \phi \\ \phi - e & e \in \phi \end{cases}$

$F_i \cong F(G-e) \cong VS^{V-2}$
or $*$



e is the only vertex of $F(G)$

not in F_1

$\mathcal{M}_e =$ forests containing $e \cong F(G_e)$

$\cong \mathcal{V} S^{V-3}$ by induction

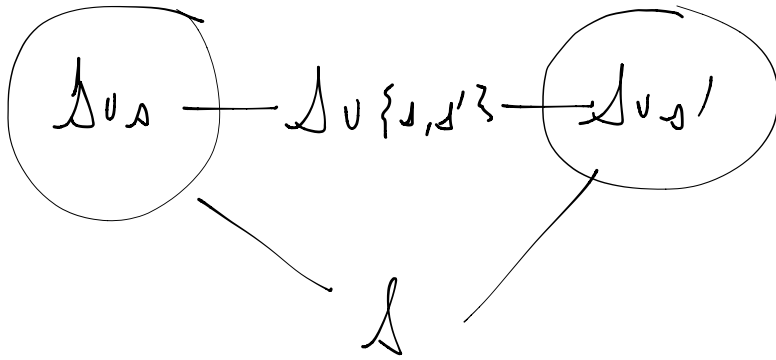
(G_e has no bridges since G has no bridges.)

So $|F(G)| \cong c(\mathcal{V} S^{V-3}) \cup_{\mathcal{V} S^{V-3}} (\mathcal{V} S^{V-2} \text{ or } *)$

$\cong \mathcal{V} S^{V-2} \checkmark$

What about $\mathcal{M}_{e>A}$?

You can add spheres to any component of $M \setminus S$ independently, you still get a sphere system
 so $\mathcal{M}_{e>A} = * S'(P)$, $P =$ component of $M \setminus S$

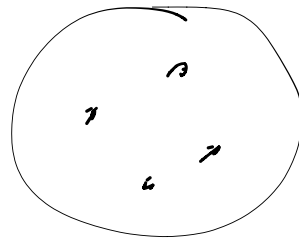


Complex of sphere systems in \mathbb{P}
 $= S'(\mathbb{P}) \cong S(\mathbb{P})$




new sphere is determined up to isotopy
 by how it separates the old spheres

Draw spheres as dots

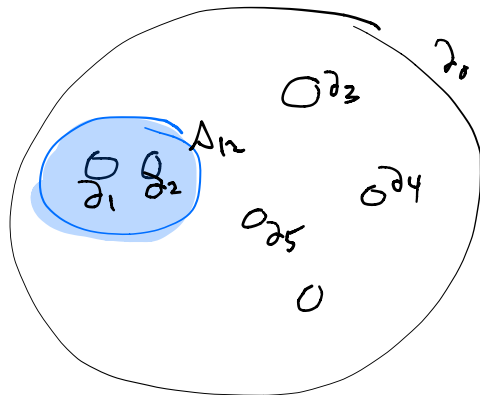


Prop $S(\mathbb{P}) \cong \mathbb{V}S^{B-4}$ $B = \# \text{bdry components of } \mathbb{P}$

pf $B=3 \Rightarrow \text{can't add spheres} \Rightarrow S(\mathbb{P}) = \emptyset$

$B=4 \Rightarrow$  3 spheres, none compatible
 $\Rightarrow S(\mathbb{P}) \cong \mathbb{V}S^0$

$B \gg 4$. Choose Δ_{12} cutting off 2 spheres ∂_1 and ∂_2



$$S_1 \subset S(\mathcal{P}) = \langle \text{spheres compatible with } \Delta_0 \rangle$$

$$\equiv \text{core on } \Delta_{12}$$

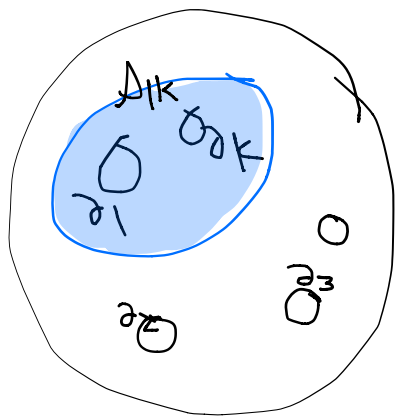
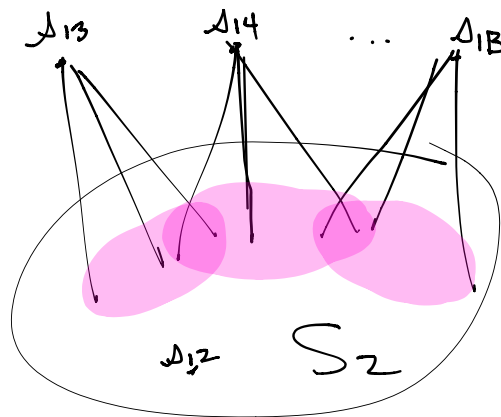
(Not in S_1 : spheres separating ∂_1 from ∂_2)

$$S_2 = \text{all spheres except } \Delta_{13}, \Delta_{14}, \dots, \Delta_{1B}$$

Claim S_2 deformation retracts to S_1 (so is contractible)

If we can prove that, we're done by induction:

$$S(P) \simeq \bigvee_{k \geq 3} \text{Susp}(\text{lk } \partial_{1k})$$



$$\text{lk}(\Delta_{1k}) = S(P')$$

P' has $B-1$ ∂ -components

$$\Rightarrow S(P') \simeq \bigvee S^{B-5}$$

$$\begin{aligned} \text{So } S(P) &\simeq \bigvee_{k \geq 3} \text{Susp}(\text{lk } \Delta_{1k}) \simeq \bigvee \text{Susp}(\bigvee S^{B-5}) \\ &\simeq \bigvee S^{B-4} \checkmark \end{aligned}$$

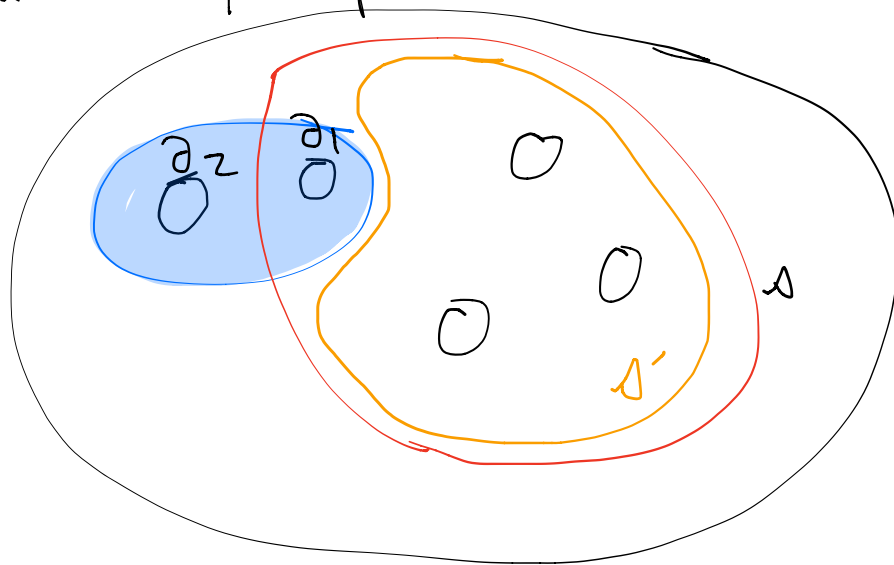
To contract $S_2 \rightarrow S_1$, define

inside of \mathcal{A} = side containing ∂_1

size of \mathcal{A} = # of ∂ components on inside

Add spheres to S_1 in order of decreasing size

Add spheres of size $B-2$



Push S_1 across ∂_1 to obtain S_1'

Check: $\text{lk } S_1 \cap S_1 = \text{core on } S_1' \simeq \text{pt}$

Exercise: What happens next? (finish the proof)

Note I "un-barycentrally subdivided"
 $S(P)$ in to abac proof.

Theorem is also true if you don't allow
spheres which separate M (\leftrightarrow separating
edges of G) but is harder to prove.

Application: $\varphi \in \text{Out}(F_n)$ gives a simplicial
automorphism of K_n

Thm The map $\text{Out}(F_n) \rightarrow \text{Aut}(K_n)$ is
an isomorphism

The first step is showing any simplicial
automorphism of K_n must send vertices
 (g, G) to (g', G') with $G' \cong G$

